ON THE DAMPING OF HOMOGENEOUS TURBULENCE IN A MAGNETIC FIELD

(O ZATUKHANII ODNORODNOI TURBULENTNOSTI V MAGNITNOM POLE)

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The damping of turbulent motion in an incompressible fluid in a magnetic field neglecting nonlinear effects was considered by Lehnert [1] (see also [2]). The damping of turbulence for small magnetic Reynolds numbers was studied in [3]. However, the analysis made in [1 and 2] is incomplete. In fact, as will be shown herein, the particular case of an initial perturbation was examined in [1]. The asymptotic behavior of the obtained solutions with time is not investigated in [1 to 3].

Let us write the linearized equations of magnetohydrodynamics for the motion of an incompressible conducting fluid in a magnetic field

$$\frac{\partial h_i}{\partial t} = \frac{c^2}{4\pi\sigma}\frac{\partial^2 h_i}{\partial x_\alpha^2} + H_0\frac{\partial v_i}{\partial x_\alpha}\lambda_\alpha, \qquad \frac{\partial v_i}{\partial x_i} = 0 \qquad \left(\boldsymbol{\lambda} = \frac{\mathbf{H}_0}{H_0}\right) \tag{1}$$

$$\frac{dv_i}{dt} = \frac{H_0}{4\pi\rho}\lambda_\alpha\frac{\partial h_i}{\partial x_\alpha} + \nu\frac{\partial^2 v_i}{\partial x_\alpha^2}, \qquad \frac{\partial h_i}{\partial x_i} = 0, \qquad \mathbf{H} = H_0\boldsymbol{\lambda} + \mathbf{h}$$

Here $\mathbf{H}_0$ is the mean magnetic field, $\mathbf{h}$ the fluctuating field, σ and ν the conductivity and kinematic viscosity of the fluid.

It has been taken into account in the equations for the velocity that in the case under consideration

$$\frac{\partial}{\partial x_i}\left(p + \frac{H_\alpha^2}{8\pi}\right) = 0$$

where p is the fluid pressure.

Let us introduce the notation

$$Q_1 = \frac{4\pi\rho}{H_0^2}\langle v_i v_j'\rangle, \qquad Q_2 = \frac{\sqrt{4\pi\rho}}{H_0^2}\langle v_i h_j' - h_i v_j'\rangle, \qquad Q_3 = \frac{1}{H_0^2}\langle h_i h_j'\rangle$$

Here the unprimed quantities are taken at the point x, and the primed quantities at the point $\mathbf{x}' = \mathbf{x} + \mathbf{r}$; the averaging is over all space. Let us also introduce nondimensional variables

$$\tau = \frac{\sigma H_0^2}{\rho c^2}t, \qquad y = \frac{\sigma H_0}{c^2}\left(\frac{4\pi}{\rho}\right)^{1/2} x$$

The mean quantities Q_α depend only on the difference $\mathbf{r} = \mathbf{x}' - \mathbf{x}$. Let us consider the vector λ to be directed along r_1. Then, as in [1], we obtain the following system for Q_α from Equations (1):

$$\left(\frac{\partial}{\partial\tau} - 2R\nabla^2\right) Q_1 = \frac{\partial Q_2}{\partial r_1}, \qquad \left(\frac{\partial}{\partial\tau} - 2\nabla^2\right) Q_3 = -\frac{\partial Q_2}{\partial r_1}$$

$$\left[\frac{\partial}{\partial\tau} - (1+R)\nabla^2\right] Q_2 = 2\frac{\partial}{\partial r_1}(Q_1 - Q_3), \qquad R = \frac{4\pi\sigma\nu}{c^2}$$

Hence, we obtain the system of equations

$$\left(\frac{\partial}{\partial\tau} + 2Rk^2\right)\Phi_1 = ik_1\Phi_2, \qquad \left(\frac{\partial}{\partial\tau} + 2k^2\right)\Phi_3 = -ik_1\Phi_2$$

$$\left(\frac{\partial}{\partial\tau} + (1+R)k^2\right)\Phi_2 = 2ik_1(\Phi_1 - \Phi_3) \tag{2}$$

for the Fourier transform

$$\Phi_\alpha(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^{3/2}\int Q_\alpha(\mathbf{r})\, e^{i\mathbf{kr}}\, d\mathbf{r}$$

The system (2) has a solution for arbitrary initial values of Φ_i at $\tau = 0$. Its solution, obtained by Lehnert (in finding the solution it was consisered that the desired functions have a time dependence of the form $e^{m\tau}$), has the property that the initial values of the functions Q_α are expressed in terms of the initial values of one of them, i.e. it is not general. Evidently the initial values of the functions Q_α (in particular, the initial values of the kinetic and magnetic energies) may be arbitrary. The solution of the system (2) for arbitrary initial values is

$$\Phi_1 = \Phi_1^\circ \frac{1}{2\beta}\left[(p_1 + 2k^2)e^{p_1\tau} - (p_2 + 2k^2)e^{p_2\tau}\right] + \frac{ik_1 e^{\alpha\tau}\Phi_2^\circ}{\beta^2}\times$$
$$\times\left[\frac{(p_1+2k^2)}{2}e^{\beta\tau} + \frac{(p_2+2k^2)}{2}e^{-\beta\tau} + k^2(R-1)\right] + \frac{2(\Phi_1^\circ + \Phi_3^\circ)}{\beta^2}k_1^2 e^{\alpha\tau}(\cosh\beta\tau - 1)$$

$$\Phi_2 = \Phi_2^\circ e^{\alpha\tau}\left[1 + \frac{4k_1^2}{\beta^2}(1 - \cosh\beta\tau)\right] + \frac{4ik_1\Phi_1^\circ}{\beta}\sinh\beta\tau e^{\alpha\tau} +$$
$$+ \frac{2ik_1}{\beta^2}(\Phi_1^\circ + \Phi_3^\circ)e^{\alpha\tau}\left[k^2(R-1) - \frac{(p_1+2Rk^2)}{2}e^{\beta\tau} - \frac{(p_2+2Rk^2)}{2}e^{-\beta\tau}\right]$$

$$\Phi_3 = \frac{\Phi_1^\circ}{2\beta}\left[(p_2 + 2Rk^2)e^{p_2\tau} - (p_1 + 2Rk^2)e^{p_1\tau}\right] +$$
$$+ \frac{ik_1\Phi_2^\circ}{\beta^2}e^{\alpha\tau}\left[k^2(R-1) - \frac{p_1 + 2Rk^2}{2}e^{\beta\tau} - \frac{p_2 + 2Rk^2}{2}e^{-\beta\tau}\right] -$$
$$- \frac{2k_1^2(\Phi_1^\circ + \Phi_3^\circ)e^{\alpha\tau}}{\beta^2}\left[1 + \frac{(p_1 + 2Rk^2)e^{\beta\tau}}{2(p_1 + 2k^2)} + \frac{(p_2 + 2Rk^2)e^{-\beta\tau}}{2(p_2 + 2k^2)}\right]$$

$$\left(p_{1,2} = -k^2(1+R) \pm \sqrt{k^4(R-1)^2 - 4k_1^2} = \alpha \pm \beta\right) \tag{3}$$

Let us investigate the solution in two particular cases: for $R = 1$ and $R \to \infty$. From (3) we obtain for $R = 1$

$$\Phi_1 = [\Phi_1^\circ \cos 2k_1\tau + {}^1/_2 i\Phi_2^\circ \sin 2k_1\tau +$$
$$+ {}^1/_2(\Phi_1^\circ + \Phi_3^\circ)(1 - \cos 2k_1\tau)]\exp(-2k^2\tau)$$

$$\Phi_2 = [i(\Phi_1^\circ - \Phi_3^\circ)\sin 2k_1\tau + \Phi_2^\circ\cos 2k_1\tau]\exp(-2k^2\tau)$$

$$\Phi_3 = [{}^1/_2(\Phi_1^\circ + \Phi_3^\circ) + {}^1/_2(\Phi_3^\circ - \Phi_1^\circ)\cos 2k_1\tau - {}^1/_2 i\Phi_2^\circ\sin 2k_1\tau]\exp(-2k^2\tau) \tag{4}$$

Let us turn from the Fourier transforms Φ_α to the functions Q_α themselves. If the system (4) is written as

$$\Phi_\alpha = f_{\alpha\beta}(\mathbf{k})\Phi_\beta^\circ$$

then the transformation to the functions Q_α will be accomplished by means of Formula

$$Q_\alpha(r,\tau)=\left(\frac{1}{2\pi}\right)^3\int_{-\infty}^{\infty}Q_\beta(\xi,0)\,d\xi\int_{-\infty}^{\infty}f_{\alpha\beta}(\mathbf{k})\,e^{i\mathbf{k}(\xi-\mathbf{r})}d\mathbf{k} \tag{5}$$

The integrals

$$\int_{-\infty}^{\infty}f_{\alpha\beta}\,e^{i\mathbf{k}(\mathbf{r}-\xi)}d\mathbf{k}$$

in (5) are taken in the principal-value sense. They reduce to three fundamental integrals

$$I_1=\int_{-\infty}^{\infty}\exp(-2k^2\tau+i\mathbf{k}\rho)\cos 2k_1\tau\,d\mathbf{k}=$$

$$=\frac{1}{2}\left(\frac{\pi}{2\tau}\right)^{3/2}\left\{\exp\left[-\frac{(2\tau-\rho_1)^2}{8\tau}\right]+\exp\left[-\frac{(2\tau+\rho_1)^2}{8\tau}\right]\right\}\exp\left(-\frac{\rho_2^2+\rho_3^2}{8\tau}\right)$$

$$I_2=i\int_{-\infty}^{\infty}\sin 2k_1\tau\exp(-2k^2\tau+i\mathbf{k}\rho)\,d\mathbf{k}=$$

$$=\frac{1}{2}\left(\frac{\pi}{2\tau}\right)^{3/2}\left\{\exp\left[-\frac{(2\tau-\rho_1)^2}{8\tau}\right]-\exp-\left[\frac{(2\tau+\rho_1)^2}{8\tau}\right]\right\}\exp\left(-\frac{\rho_2^2+\rho_3^2}{8\tau}\right)$$

$$I_3=\int_{-\infty}^{\infty}\exp(-2k^2\tau+i\mathbf{k}\rho)\,d\mathbf{k}=\left(\frac{\pi}{2\tau}\right)^{3/2}\exp\left(-\frac{\rho_1^2+\rho_2^2+\rho_3^2}{8\tau}\right)$$

Investigating the asymptotic behavior of the functions Q_α as $\tau\to\infty$ it is necessary to take into account that

$$\int_{-\infty}^{\infty}Q_\alpha(\xi,t)\,d\xi=0$$

because of the condition of solenoidality of the considered tensors. For $t\to\infty$

$$I_1=\left(\frac{\pi}{2\tau}\right)^{3/2}e^{-1/2\tau}\cosh{}^1/_2\rho_1,\qquad I_2=-\left(\frac{\pi}{2\tau}\right)^{3/2}e^{-1/2\tau}\sinh{}^1/_2\,\rho_1$$

$$I_3=\left(\frac{\pi}{2\tau}\right)^{3/2}\left(1-\frac{\rho_1^2+\rho_2^2+\rho_3^2}{8\tau}\right)$$

However, the first member in I_3 vanishes upon integration with respect to ξ in (5), and it is necessary to take account only of the second member in estimating the asymptotic of Q_α. Therefore, the members with $\cos 2k_1\tau$ and $\sin 2k_1\tau$ yield exponentially decreasing integrals, and the remaining terms in (4) yield contributions, decreasing as $t^{-5/2}$, to the integrals. The correlations between the velocity and the magnetic field decrease exponentially, and besides, the contribution of $Q_2(\mathbf{r},0)$ to $Q_1(\mathbf{r},\tau)$ and $Q_2(\mathbf{r},\tau)$ also drops exponentially. Hence, for $R=1$ when the magnetic and kinetic viscosities are equal, the correlations between the velocity and the magnetic field decrease more rapidly than the other correlations; hence, the magnetic and kinetic energies drop asymptotically as $t^{-5/2}$. It is interesting that as $\tau\to\infty$

$$\Phi_1={}^1/_2\,(\Phi_1^\circ+\Phi_3^\circ)\exp(-2k^2\tau),\qquad \Phi_3=\Phi_1$$

i.e. the kinetic and magnetic energies decrease identically, albeit with an amplitude proportional to the arithmetic mean of the initial values of the magnetic and kinetic energies.

For $R\to\infty$ we separate the domain $k\gg 1/R$ out of $\mathbf{k}$-space. In the domain $k\gg 1/R$ we can represent p_1 and p_2 approximately by means of Expressions

$$p_1=-2k^2-2k_1^2/k^2R,\qquad p_2=-2k^2R+2k_1^2/k^2R$$

and, by retaining the fundamental terms of the expression, to write the system (3) as

$$\Phi_1 = \frac{\Phi_3^\circ k_1^2 e^{-2k^2\tau}}{k^4 R^2}, \qquad \Phi_2 = -\frac{2ik_1\Phi_3^\circ e^{-2k^2\tau}}{k^2 R}, \qquad \Phi_3 = \Phi_3^\circ e^{-2k^2\tau} \ (\tau \gg 1) \qquad (6)$$

Let us consider the condition $R \gg \tau \gg 1$ to be satisfied. Then the contribution of the portion of **k**-space outside the domain $k \gg 1/R$ tends to zero as $R \to \infty$ and, upon transforming to the functions Q_α themselves, the appropriate integration may be extended over all **k**-space. As follows from (6), the fundamental contribution to all the correlations is yielded by the initial correlations between the magnetic field components. Namely, the magnetic energy is predominant in the last stage of damping of the turbulence. The kinetic energy is on the order of $1/R^2$ relative to the magnetic energy, and the correlation between the magnetic field and the velocity is on the order of $1/R$. The magnetic energy decreases asymptotically as $t^{-5/2}$, just as does the correlation between the velocity and the magnetic field. In fact, the appropriate integrals for Q_3 in (5) may be reduced to an integral of the type

$$\int_0^\infty dk \int_0^{1/2\pi} d\theta \int_0^{1/2\pi} d\varphi k \sin kz \sin\theta \exp(-2k^2\tau) =$$

$$= \frac{z}{4\tau}\int_0^\infty dk \int_0^{1/2\pi} d\theta \int_0^{1/2\pi} d\varphi \cos kz \exp(-2k^2\tau)\sin\theta$$

with the asymptotic $\tau^{-5/2}$. However, the asymptotic value of the magnetic energy does not contain the factor $1/R$, and therefore, it has the greatest order of magnitude in the final stage of damping of turbulence in an ideally conducting fluid.

The damping of turbulaence in a weakly conducting fluid at low magnetic Reynolds numbers was considered in [3], where it has been found that the energy spectral density decreases as

$$\exp\left\{-2\left[\nu k^2 + \frac{(k_\alpha\lambda_\alpha)^2}{k^2}\frac{\rho c^2}{\sigma H_0^2}\right]t\right\}$$

Fluctuations of the electric field were not taken into account in that paper. However, it can be shown that taking account of these fluctuations does not change the obtained results. Indeed, from the magnetohydrodynamic equations for a weakly conducting fluid

$$\frac{\partial v_i}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{[\mathbf{j}\times\mathbf{H}_0]_i}{\rho c} + \nu\frac{\partial^2 v_i}{\partial x_\alpha^2}$$

$$\mathbf{j} = \sigma\left(\nabla U + \frac{\mathbf{v}\times\mathbf{H}_0}{c}\right), \qquad \operatorname{div}\mathbf{j} = 0, \qquad \operatorname{div}\mathbf{v} = 0$$

(here U is the electric potential, $\mathbf{H}_0$ the external field), we obtain the equations for the correlation functions

$$\frac{\partial Q_{ij}}{\partial t} = \frac{\partial L_j}{\partial r_i} - \frac{\partial l_i}{\partial r_j} + \frac{\sigma H_0}{\rho c}\left(\varepsilon_{j\alpha\beta}\lambda_\beta\frac{\partial m_i}{\partial r_\alpha} - \varepsilon_{i\alpha\beta}\lambda_\beta\frac{\partial M_j}{\partial r_\alpha}\right) + \frac{\sigma H_0^2}{\rho c^2}(\lambda_\alpha\lambda_i Q_{\alpha j} + $$
$$+ \lambda_\alpha\lambda_j Q_{i\alpha} - 2Q_{ij}) + 2\nu\frac{\partial^2 Q_{ij}}{\partial r_\alpha^2}$$

$$\frac{\partial^2 M_i}{\partial r_\alpha^2} - \frac{H_0}{c}\varepsilon_{\alpha\gamma\beta}\lambda_\beta\frac{\partial Q_{\gamma i}}{\partial r_\alpha} = \frac{\partial^2 m_i}{\partial r_\alpha^2} + \frac{H_0}{c}\varepsilon_{\alpha\gamma\beta}\lambda_\beta\frac{\partial Q_{i\gamma}}{\partial r_\alpha} = 0$$

$$\frac{\partial^2 L_j}{\partial r_\alpha^2} + \frac{\sigma H_0^2}{\rho c^2}\lambda_\alpha\lambda_i\frac{\partial Q_{\alpha j}}{\partial r_i} = -\frac{\partial^2 l_i}{\partial r_\alpha^2} + \frac{\sigma H_0^2}{\rho c^2}\lambda_\alpha\lambda_j\frac{\partial Q_{i\alpha}}{\partial r_j} = 0$$

$$\frac{\partial Q_{ij}}{\partial r_j} = \frac{\partial Q_{ij}}{\partial r_i} = 0$$

$$Q_{ij} = Q_1 = \langle v_i v_j'\rangle, \quad L_i = \left\langle\frac{p v_i'}{\rho}\right\rangle, \quad l_i = \left\langle\frac{p' v_i}{\rho}\right\rangle, \quad M_i = \langle U v_i'\rangle, \quad m_i = \langle U' v_i\rangle$$

Transforming to the Fourier representation and eliminating the pressure and potential correlations, we have just as in [3]

$$\frac{\partial \Phi_1}{\partial t} = -2\left(\nu k^2 + \frac{k_1^2}{k^2}\,\frac{\rho c^2}{\sigma H_0^2}\right)\Phi_1 \tag{7}$$

Expression (7) follows from (1) if the term $\partial h_1/\partial t$ is neglected. From (2) we then obtain

$$\frac{\partial \Phi_1}{\partial t} = -2\left(Rk^2 + \frac{k_1^2}{(1+R)\,k^2 + k_1^2 / k^2}\right) \tag{8}$$

Large values of k and small values of R correspond to low magnetic Reynolds numbers. In this approximation (7) and (8) agree. Integrating (7) and transforming to Q_1, we obtain

$$Q_1(\mathbf{r}, t) = \left(\frac{1}{2\pi}\right)^3 \int\limits_{-\infty}^{\infty} Q_1(\xi, 0)\, d\xi \int\limits_{-\infty}^{\infty} \exp\left[-2\left(\nu k^2 + \frac{k_1^2 \rho c^2}{k^2 \sigma H_0^2}\right)t + i\mathbf{k}(\xi - \mathbf{r})\right] d\mathbf{k} \tag{9}$$

The second integral in (9) may be transformed as follows:

$$8\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty \exp\left[-2\left(\nu k^2 + \frac{k_1^2\rho c^2}{k^2\sigma H_0^2}\right)t\right]\prod_{\alpha=1}^{3}\cos k_\alpha \rho_\alpha\, dk_1\, dk_2\, dk_3 =$$

$$= \int\limits_0^\infty dk \int\limits_0^{1/2\pi} d\theta \int\limits_0^{1/2\pi} d\varphi \sum_{\alpha=1}^{8} \exp\left(-2\nu k^2 t - \frac{2\rho c^2 t\cos^2\theta}{\sigma H_0^2} + i f_\alpha\right) k^2 \sin\theta =$$

$$= \frac{\sqrt{\pi}}{8\sqrt{2}}\,\frac{1}{(\nu t)^{3/2}} \int\limits_0^{1/2\pi} d\theta \int\limits_0^{1/2\pi} d\varphi \sin\theta \sum_{\alpha=1}^{8} F\left(\frac{3}{2}, \frac{1}{2}, -\frac{f_\alpha^2}{8t}\right)\exp\left(-\frac{2\rho c^2 t\cos^2\theta}{\sigma H_0^2}\right) \tag{10}$$

Here F is the degenerate hypergeometric function

$$F(\alpha, \beta, z) = 1 + \frac{\alpha}{\beta}\,\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\,\frac{z^2}{2!} + \cdots, \qquad \rho = \xi - \mathbf{r}$$

and the functions f_α are independent of time.

Let us expand the function F in a series and let us limit ourselves to the first two terms as $t \to \infty$. The first member of the expansion vanishes upon integration with respect to ξ because of the solenoidality of Q_1, and the second yields the damping law $Q_1 \sim t^{-3}$. Let us note that taking account of the pressure fluctuations yields a power law of correlation damping, and not an exponential law. As in conventional hydrodynamics [4], the asymptotic law of the degeneration of turbulence in a weakly conducting fluid turns out to be universal (independent of the initial conditions).

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